Issue XXI: Super Type System

М.Е. Сохацький [1]

[1] Національний технічний університет України
ім. Ігоря Сікорського
26 листопада 2025

Аннотація

Зміст

1 Introduction to Urs

2 Super Type System

3 Bosonic Modality

The $\bigcirc$ modality in cohesive type theory projects a type to bosonic parity ($g = 0$). For a type $A : \mathbf{U}_{i,g}$, $\bigcirc A$ forces the type to be bosonic, aligning with supergeometry and quantum physics.

In Urs, $\bigcirc$ operates on types in graded universes from **Graded**, with applications in bosonic quantum fields **qubit** and supergeometry **SmthSet**.

3.1 Bose

Definition 1 (Bosonic Modality Formation). The $\bigcirc$ modality is a type operator on graded universes, mapping to bosonic parity:

$$\bigcirc : \prod_{i:\mathbb{N}} \prod_{g:\mathbf{Grade}} \mathbf{U}_{i,g} \to \mathbf{U}_{i,0}.$$

```
def bosonic (i : Nat) (g : Grade) (A : U i g) : U i 0
```

Definition 2 (Bosonic Modality Introduction). Applying $\bigcirc$ to a type A produces $\bigcirc A$ with bosonic parity:

$$\Gamma \vdash A : \mathbf{U}_{i,g} \quad \to \quad \Gamma \vdash \bigcirc A : \mathbf{U}_{i,0}.$$

Definition 3 (Bosonic Modality Elimination). The eliminator for $\bigcirc A$ maps bosonic types to properties in $\mathbf{U_0}$:

$$\mathbf{Ind}_{\bigcirc} : \prod_{i:\mathbb{N}} \prod_{g:\mathbf{Grade}} \prod_{A:\mathbf{U}_{i,g}} \prod_{\phi:(\bigcirc A)\to \mathbf{U_0}} \left(\prod_{a:\bigcirc A} \phi\ a \right) \to \prod_{a:\bigcirc A} \phi\ a.$$

```
def bosonic_ind (i : Nat) (g : Grade) (A : U i g)
    (phi : (bosonic i g A) -> U_0)
    (h : Π (a : bosonic i g A), phi a)
  : Π (a : bosonic i g A), phi a
```

Theorem 1 (Idempotence of Bosonic). The $\bigcirc$ modality is idempotent, as it always projects to bosonic parity:

$$\bigcirc\textbf{-idem} : \prod_{i:\mathbb{N}} \prod_{g:\mathbf{Grade}} \prod_{A:\mathbf{U}_{i,g}} (\bigcirc(\bigcirc A)) = (\bigcirc A).$$

```
def bosonic_idem (i : Nat) (g : Grade) (A : U i g)
  : (bosonic i 0 (bosonic i g A)) = (bosonic i g A)
```

Theorem 2 (Bosonic Qubits). For $C, H : \mathbf{U_0}$, the type $\bigcirc\mathbf{Qubit}(C, H)$ models bosonic quantum states:

$$\bigcirc\textbf{-qubit} : \prod_{i:\mathbb{N}} \prod_{g:\mathbf{Grade}} \prod_{C,H:\mathbf{U_0}} (\bigcirc\mathbf{Qubit}(C, H)) : \mathbf{U}_{i,0}.$$

```
def bosonic_qubit (i : Nat) (g : Grade) (C H : U_0) : U i 0
 := bosonic i g (Qubit C H)
```

3.2 Braid

The $\mathbf{Braid}_n(X)$ type models the braid group $B_n(X)$ on n strands over a smooth set $X : \mathbf{SmthSet}$, the fundamental group of the configuration space $\mathbf{Conf}^n(X)$, used in knot theory, quantum computing, and smooth geometry.

In Urs, $\mathbf{Braid}_n(X)$ is a type in $\mathbf{U_0}$, parameterized by $n : \mathbf{Nat}$ and $X : \mathbf{SmthSet}$, supporting braid generators σ_i and relations, with applications to anyonic quantum gates and knot invariants.

Definition 4 (Braid Formation). The type $\mathbf{Braid}_n(X)$ is formed for each $n : \mathbf{Nat}$ and $X : \mathbf{SmthSet}$:

$$\mathbf{Braid} : \prod_{n:\mathbf{Nat}} \prod_{X:\mathbf{SmthSet}} \mathbf{U_0}.$$

```
def Braid (n : Nat) (X : SmthSet) : U_0
(* Braid group type *)
```

Definition 5 (Braid Introduction). Terms of type $\mathbf{Braid}_n(X)$ are introduced via the **braid** constructor, representing generators σ_i for $i : \mathbf{Fin}\ (n-1)$:

$$\mathbf{braid} : \prod_{n:\mathbf{Nat}} \prod_{X:\mathbf{SmthSet}} \prod_{i:\mathbf{Fin}\ (n-1)} \mathbf{Braid}_n(X).$$

```
def braid (n : Nat) (X : SmthSet) (i : Fin (n−1)) : Braid n X
(* Braid generator sigma_i *)
```

Definition 6 (Braid Elimination). The eliminator for $\mathbf{Braid}_n(X)$ maps braid elements to properties in $\mathbf{U_0}$:

$$\mathbf{BraidInd} : \prod_{n:\mathbf{Nat}} \prod_{X:\mathbf{SmthSet}} \prod_{\beta:\mathbf{Braid}_n(X)\to\mathbf{U_0}} \left(\prod_{b:\mathbf{Braid}_n(X)} \beta\ b \right) \to \prod_{b:\mathbf{Braid}_n(X)} \beta\ b.$$

```
def braid_ind (n : Nat) (X : SmthSet)
     (beta : Braid n X −> U_0)
     (h : Π (b : Braid n X), beta b)
   : Π (b : Braid n X), beta b
```

Theorem 3 (Braid Relations). For $n : \mathbf{Nat}$, $X : \mathbf{SmthSet}$, $\mathbf{Braid}_n(X)$ satisfies the braid group relations (Commutation and Yang-Baxter):

$$\prod_{n:\mathbf{Nat}} \prod_{X:\mathbf{SmthSet}} \prod_{i,j:\mathbf{Fin}\ (n-1),\ |i-j|\geq 2} \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i,$$

$$\prod_{n:\mathbf{Nat}} \prod_{X:\mathbf{SmthSet}} \prod_{i:\mathbf{Fin}\ (n-2)} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}.$$

```
def braid_rel_comm (n : Nat) (X : SmthSet) (i j : Fin (n−1))
  : Path (braid i · braid j) (braid j · braid i)

def braid_rel_yang (n : Nat) (X : SmthSet) (i : Fin (n−2))
  : Path (braid i · braid (i+1) · braid i) (braid (i+1) · braid i ·
braid (i+1))
```

Theorem 4 (Configuration Space Link). For $n : \mathbf{Nat}$, $X : \mathbf{SmthSet}$, $\mathbf{Braid}_n(X)$ is the fundamental groupoid of $\mathbf{Conf}^n(X)$:

$$\prod_{n:\mathbf{Nat}} \prod_{X:\mathbf{SmthSet}} \mathbf{Braid}_n(X) \cong \pi_1(\mathbf{Conf}^n(X)).$$

```
def braid_conf (n : Nat) (X : SmthSet)
  : Path (Braid n X) (pi_1 (Conf n X))
```

Theorem 5 (Quantum Braiding). For $C, H : \mathbf{U_0}$, $\mathbf{Braid}_n(X)$ acts on $\mathbf{Qubit}(C,H)^{\otimes n}$ as braiding operators:

$$\mathbf{braid\_qubit} : \prod_{n:\mathbf{Nat}} \prod_{C,H:\mathbf{U_0}} \prod_{X:\mathbf{SmthSet}} \mathbf{Braid}_n(X) \to \left(\mathbf{Qubit}(C,H)^{\otimes n} \to \mathbf{Qubit}(C,H)^{\otimes n}\right).$$

```
def braid_qubit (n : Nat) (C H : U_0) (X : SmthSet)
  : Braid n X -> (Qubit C H)^n -> (Qubit C H)^n
```

Theorem 6 (Braid Group Delooping). For $n : \mathbf{Nat}$, the delooping $\mathbf{BB}_n$ of the braid group B_n is a 1-groupoid:

$$\mathbf{BB}_n : \mathbf{Grpd}\ 1 \equiv \Im(\mathbf{Conf}^n(\mathbb{R}^2)).$$

```
def BB_n (n : Nat) : Grpd 1 := ℑ (Conf n ℝ^2)
```

3.3 Graded Universes

Graded Universes. The $\mathbf{U}_\alpha$ type represents a graded universe indexed by a monoid $\mathcal{G} = \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$, where $\alpha \in \mathcal{G}$ encodes a level ($\mathbb{N}$) and parity ($\mathbb{Z}/2\mathbb{Z}$: $0 =$ bosonic, $1 =$ fermionic). Graded universes support type hierarchies with cumulativity, graded tensor products, and coherence rules, used in supergeometry (e.g., bosonic/fermionic types), quantum systems (e.g., graded qubits), and cohesive type theory.

In Urs, $\mathbf{U}_\alpha$ is a type indexed by $\alpha : \mathcal{G}$, with operations like lifting, product formation, and graded tensor products, extending standard universe hierarchies to include parity, building on **Tensor**.

Definition 7 (Grading Monoid). The grading monoid $\mathcal{G}$ is defined as $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$, with operation $\oplus$ and neutral element $\mathbf{0}$, encoding level and parity.

$$
\begin{aligned}
\mathcal{G} : \mathbf{Type} &\equiv \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\
\oplus : \mathcal{G} &\to \mathcal{G} \to \mathcal{G}, \\
(\alpha, \beta) &\mapsto (\text{fst } \alpha + \text{fst } \beta,\ (\text{snd } \alpha + \text{snd } \beta) \mod 2), \\
\mathbf{0} : \mathcal{G} &\equiv (0, 0).
\end{aligned}
$$

```
def 𝒢 : Type := ℕ × ℤ/2ℤ
def ⊕ (α β : 𝒢) : 𝒢 := (fst α + fst β, (snd α + snd β) mod 2)
def ⊬ : 𝒢 := (0, 0)
```

Definition 8 (Graded Universe Formation). The universe $\mathbf{U}_\alpha$ is a type indexed by $\alpha : \mathcal{G}$, containing types of grade α. A shorthand notation $\mathbf{U}_{i,g}$ is used for $\mathbf{U}\ (i, g)$.

$$
\begin{aligned}
\mathbf{U} : \mathcal{G} &\to \mathbf{Type}, \\
\mathbf{Grade} : \mathbf{Set} &\equiv \{0, 1\}, \\
\mathbf{U}_{i,g} : \mathbf{Type} &\equiv \mathbf{U}(i, g) : \mathbf{U}_{i+1}.
\end{aligned}
$$

```
def U (α : 𝒢) : Type := Universe α
def Grade : Set := {0, 1}
def U (i : Nat) (g : Grade) : Type := U (i, g)
def U_0 (g : Grade) : U (1, g) := U (0, g)
def U_00 : Type := U (0, 0)
def U_10 : Type := U (1, 0)
def U_01 : Type := U (0, 1)
```

Definition 9 (Graded Universe Coherence Rules). Graded universes support coherence rules for lifting, product formation, and substitution, ensuring typetheoretic consistency.

$$\begin{aligned}
\textbf{lift} &: \prod_{\alpha,\beta:\mathcal{G}} \prod_{\delta:\mathcal{G}} \mathbf{U}\ \alpha \to (\beta = \alpha \oplus \delta) \to \mathbf{U}\ \beta, \\
\textbf{univ} &: \prod_{\alpha:\mathcal{G}} \mathbf{U}\ (\alpha \oplus (1,0)), \\
\textbf{cumul} &: \prod_{\alpha,\beta:\mathcal{G}} \prod_{A:\mathbf{U}\ \alpha} \prod_{\delta:\mathcal{G}} (\beta = \alpha \oplus \delta) \to \mathbf{U}\ \beta, \\
\textbf{prod} &: \prod_{\alpha,\beta:\mathcal{G}} \prod_{A:\mathbf{U}\ \alpha} \prod_{B:A\to\mathbf{U}\ \beta} \mathbf{U}\ (\alpha \oplus \beta), \\
\textbf{subst} &: \prod_{\alpha,\beta:\mathcal{G}} \prod_{A:\mathbf{U}\ \alpha} \prod_{B:A\to\mathbf{U}\ \beta} \prod_{t:A} \mathbf{U}\ \beta, \\
\textbf{shift} &: \prod_{\alpha,\delta:\mathcal{G}} \prod_{A:\mathbf{U}\ \alpha} \mathbf{U}\ (\alpha \oplus \delta).
\end{aligned}$$

```
def lift (α β : 𝒢) (δ : 𝒢) (e : U α) : β = α ⊕ δ → U β :=
  λ eq : β = α ⊕ δ, transport (λ x : 𝒢, U x) eq e

def univ-type (α : 𝒢) : U (α ⊕ (1, 0)) :=
  lift α (α ⊕ (1, 0)) (1, 0) (U α) refl

def cumul (α β : 𝒢) (A : U α) (δ : 𝒢) : β = α ⊕ δ → U β :=
  lift α β δ A

def prod-rule (α β : 𝒢) (A : U α) (B : A → U β) : U (α ⊕ β) :=
  Π (x : A), B x

def subst-rule (α β : 𝒢) (A : U α) (B : A → U β) (t : A) : U β :=
  B t

def shift (α δ : 𝒢) (A : U α) : U (α ⊕ δ) :=
  lift α (α ⊕ δ) δ A refl
```

Definition 10 (Graded Tensor Introduction). Graded tensor products combine types with matching levels, combining parities.

$$\begin{aligned}
\textbf{tensor} &: \prod_{i:\mathbb{N}} \prod_{g_1,g_2:\textbf{Grade}} \mathbf{U}_{i,g_1} \to \mathbf{U}_{i,g_2} \to \mathbf{U}_{i,(g_1+g_2 \bmod 2)}, \\
\textbf{pair-tensor} &: \prod_{i:\mathbb{N}} \prod_{g_1,g_2:\textbf{Grade}} \prod_{A:\mathbf{U}_{i,g_1}} \prod_{B:\mathbf{U}_{i,g_2}} \prod_{a:A} \prod_{b:B} \textbf{tensor}(i, g_1, g_2, A, B).
\end{aligned}$$

```
def tensor (i : Nat) (g₁ g₂ : Grade)
    (A : U i g₁) (B : U i g₂) : U i (g₁ + g₂ mod 2)
 := A ⊗ B

def pair-tensor (i : Nat) (g₁ g₂ : Grade) (A : U i g₁)
    (B : U i g₂) (a : A) (b : B) : tensor i g₁ g₂ A B
 := a ⊗ b
```

Definition 11 (Graded Tensor Eliminators). Eliminators for graded tensor products project to their components.

$$\otimes\text{-}\mathbf{prj}_1 : (A \otimes B) \to A,$$
$$\otimes\text{-}\mathbf{prj}_2 : (A \otimes B) \to B.$$

```
def pr1 (i : Nat) (g1 g2 : Grade)
    (A : U i g1) (B : U i g2) (p : A ⊗ B) : A := p.1

def pr2 (i : Nat) (g1 g2 : Grade)
    (A : U i g1) (B : U i g2) (p : A ⊗ B) : B := p.2
```

Theorem 7 (Monoid Properties). The grading monoid $\mathcal{G}$ satisfies associativity and identity laws.

$$\begin{aligned} \textbf{assoc} &: ((\alpha \oplus \beta) \oplus \gamma) = (\alpha \oplus (\beta \oplus \gamma)), \\ \textbf{id-left} &: (\alpha \oplus \mathbf{0}) = \alpha, \\ \textbf{id-right} &: (\mathbf{0} \oplus \alpha) = \alpha. \end{aligned}$$

```
def assoc (α β γ : 𝒢) : (α ⊕ β) ⊕ γ = α ⊕ (β ⊕ γ) := refl
def ident−left (α : 𝒢) : α ⊕ ⊬ = α := refl
def ident−right (α : 𝒢) : ⊬ ⊕ α = α := refl
```