Monads and Descent

Jean Bénabou and Jacques Roubaud Communicated by Henri Cartan

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Анотація

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

1 Chevalley Bifibrations and Descent

Let $P : \mathbf{M} \to \mathbf{A}$ denote a bifibrant functor [1]. For an object $A \in \mathbf{A}$, let $\mathbf{M}(A)$ denote the fibre over A. We assume that \mathbf{A} has fibred products.

1.1 Monad Associated with an Arrow

Let $\mathfrak{a}: A_1 \to A_0$ be an arrow in **A**. Denote by

$$a^*: \mathbf{M}(A_0) \to \mathbf{M}(A_1) \quad [\text{resp. } a_*: \mathbf{M}(A_1) \to \mathbf{M}(A_0)]$$

the inverse image functor (resp. direct image functor), and

 $\eta^{\mathfrak{a}}: \mathrm{Id}_{\mathbf{M}(A_1)} \to \mathfrak{a}^*\mathfrak{a}_*; \quad \mathfrak{e}^{\mathfrak{a}}: \mathfrak{a}_*\mathfrak{a}^* \to \mathrm{Id}_{\mathbf{M}(A_0)}$

the canonical natural transformations making a_* a left adjoint to a^* . This adjunction defines [2] on $\mathbf{M}(A_1)$ the monad $\mathbf{T}^{\alpha} = (\mathbf{T}^{\alpha}, \mu^{\alpha}, \eta^{\alpha})$, where

$$T^{\mathfrak{a}} = \mathfrak{a}^* \mathfrak{a}_* : \mathbf{M}(A_1) \to \mathbf{M}(A_1), \quad \mu^{\mathfrak{a}} = \mathfrak{a}^* \varepsilon^{\mathfrak{a}} \mathfrak{a}_* : T^{\mathfrak{a}} \circ T^{\mathfrak{a}} \to T^{\mathfrak{a}}.$$

Let $M^{\mathfrak{a}}$ denote the category $M(A_1)^{(\mathsf{T}^{\mathfrak{a}})}$ of algebras over the monad $\mathsf{T}^{\mathfrak{a}},$ and let

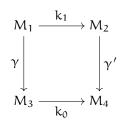
 $\mathbf{U}^{\mathsf{T}^{\mathfrak{a}}}: \mathbf{M}^{\mathfrak{a}} \to \mathbf{M}(A_1), \quad \Phi^{\mathfrak{a}}: \mathbf{M}(A_0) \to \mathbf{M}^{\mathfrak{a}}$

be the canonical functors.

1.2 Chevalley Property

Definition 1. The functor P is a *Chevalley functor* if it satisfies the following property (C):

(C) For every commutative diagram in M



whose image under P is a cartesian square in \mathbf{A} , if γ and γ' are cartesian and k_0 is cocartesian, then k_1 is cocartesian.

1.3 Characterization of Descent Data

Assume henceforth that $P: \mathbf{M} \to \mathbf{A}$ is a Chevalley functor. Let $\mathbf{a}: A_1 \to A_0$ be an arrow in \mathbf{A} . Let A_2 be the fibred product $A_1 \times_{A_0} A_1$, with canonical projections $\mathbf{a}_1, \mathbf{a}_2: A_2 \to A_1$. The property (C) defines, for every object $M_1 \in \mathbf{M}(A_1)$, a canonical bijection, natural in M_1 ,

$$\operatorname{Hom}_{\boldsymbol{M}(A_2)}(\mathfrak{a}_1^*(M_1),\mathfrak{a}_2^*(M_1))\to\operatorname{Hom}_{\boldsymbol{M}(A_1)}(\mathsf{T}^{\mathfrak{a}}(M_1),M_1),$$

denoted $\phi \mapsto K^{\mathfrak{a}}(\phi)$.

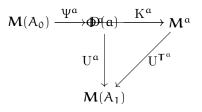
Lemma 1. An arrow $\varphi : \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1)$ such that $P(\varphi) = id_{A_2}$ is a descent datum if and only if $K^{\mathfrak{a}}(\varphi)$ is an algebra over the monad $T^{\mathfrak{a}}$.

Let D(a) denote the category of descent data relative to a, and let

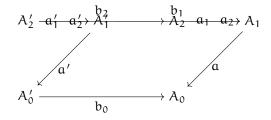
$$\Psi^{\mathfrak{a}}: \mathbf{M}(\mathcal{A}_0) \to \mathcal{D}(\mathfrak{a}), \quad \mathcal{U}^{\mathfrak{a}}: \mathcal{D}(\mathfrak{a}) \to \mathbf{M}(\mathcal{A}_1)$$

be the canonical functors.

Theorem 1. The correspondence $\varphi \mapsto K^{\mathfrak{a}}(\varphi)$ induces an equivalence of categories $K^{\mathfrak{a}}: D(\mathfrak{a}) \to M^{\mathfrak{a}}$, making the following diagram commute:



Proposition 1. The correspondence $\phi \mapsto K^{\alpha}(\phi)$ is universal. Precisely, for an arrow $b_0: A'_0 \to A_0$ in A, consider the change-of-base diagram in A:



For $M_1 \in \mathbf{M}(A_1)$ and $\varphi : \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1)$ in $\mathbf{M}(A_2)$,

$$\mathbf{K}^{\mathfrak{a}'}(\mathfrak{b}_2^*(\varphi)) = \mathfrak{b}_1^*(\mathbf{K}^{\mathfrak{a}}(\varphi))$$

In particular, taking $A'_0 = A_1$ and $b_0 = a$, if φ is a descent datum, then $b_2^*(\varphi)$ is an effective descent datum. The converse holds, yielding:

Corollary 1. An arrow $\varphi : \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1) \in \mathbf{M}(A_2)$ is a descent datum if and only if its inverse image $\mathfrak{b}_2^*(\varphi)$ under the canonical change of base $\mathfrak{b}_0 = \mathfrak{a} : A_0' = A_1 \to A_0$ is an effective descent datum.

This eliminates the need for the "cocycle condition" in subsequent arguments.

2 First Applications

Using Theorem 1, Beck's criterion [2] provides necessary and sufficient conditions for Ψ^{α} to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by a^* .

Proposition 2. If cokernels of pairs of arrows exist in $\mathbf{M}(A_0)$, then Ψ^{α} has a left adjoint.

Proposition 3. The functor $\Psi^{\mathfrak{a}}$ is faithful if and only if \mathfrak{a}^* is faithful.

Proposition 4. If a^* reflects cokernels, then Ψ^a is fully faithful. In particular, if all fibres of M are abelian, then

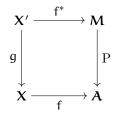
 $\Psi^{\mathfrak{a}}$ faithful $\iff \Psi^{\mathfrak{a}}$ fully faithful $\iff \mathfrak{a}^{*}$ faithful.

Definition 2. An arrow $a : A_1 \to A_0$ is *faithfully flat* if a^* commutes with cokernels and reflects isomorphisms.

Proposition 5. If $a : A_1 \to A_0$ is faithfully flat and cokernels exist in $\mathbf{M}(A_0)$, then Ψ^a is an equivalence of categories.

3 First Examples of Chevalley Functors

- 1. If **A** is the dual of the category of commutative rings and **M** is the dual of the category of modules over varying commutative rings, the obvious functor $P: \mathbf{M} \to \mathbf{A}$ is Chevalley.
- 2. If **A** is a category with fibred products and $\mathbf{M} = \mathsf{Fl}(\mathbf{A})$ is the category of arrows in **A**, the "target" functor $P : \mathbf{M} \to \mathbf{A}$ is Chevalley.
- 3. If $P: \mathbf{M} \to \mathbf{A}$ and $Q: \mathbf{N} \to \mathbf{M}$ are Chevalley, their composite $P \circ Q$ is Chevalley.
- 4. If $P: M \to A$ is Chevalley and I is any category, the functor $P^I: M^I \to A^I$ is Chevalley.
- 5. In a cartesian diagram of categories



if X has fibred products, f preserves fibred products, and P is Chevalley, then $f^*(P)$ is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether Ψ^{α} is faithful, fully faithful, or an equivalence when the fibres of **M** are algebraic categories (e.g., categories of modules).

Література

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- [3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).