Monads and Descent

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Анотація

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

1 Chevalley Bifibrations and Descent

Let $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ denote a bifibrant functor [1]. For an object $A \in \mathbf{A}$, let $\mathbf{M}(A)$ denote the fibre over A. We assume that $\mathbf{A}$ has fibred products.

1.1 Monad Associated with an Arrow

Let $a : A_1 \to A_0$ be an arrow in $\mathbf{A}$. Denote by

$$a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1) \quad [\text{resp. } a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)]$$

the inverse image functor (resp. direct image functor), and

$$\eta^a : \mathrm{Id}_{\mathbf{M}(A_1)} \to a^* a_*; \quad \varepsilon^a : a_* a^* \to \mathrm{Id}_{\mathbf{M}(A_0)}$$

the canonical natural transformations making a_* a left adjoint to a^*. This adjunction defines [2] on $\mathbf{M}(A_1)$ the monad $\mathbf{T}^a = (T^a, \mu^a, \eta^a)$, where

$$T^a = a^* a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_1), \quad \mu^a = a^* \varepsilon^a a_* : T^a \circ T^a \to T^a.$$

Let $\mathbf{M}^a$ denote the category $\mathbf{M}(A_1)^{(\mathbf{T}^a)}$ of algebras over the monad $\mathbf{T}^a$, and let

$$\mathrm{U}^{\mathbf{T}^a} : \mathbf{M}^a \to \mathbf{M}(A_1), \quad \Phi^a : \mathbf{M}(A_0) \to \mathbf{M}^a$$

be the canonical functors.

1.2 Chevalley Property

Definition 1. The functor P is a *Chevalley functor* if it satisfies the following property (C):

(C) For every commutative diagram in $\mathbf{M}$

$$\begin{tikzcd}
M_1 \arrow[r, "k_1"] \arrow[d, "\gamma"'] & M_2 \arrow[d, "\gamma'"] \\
M_3 \arrow[r, "k_0"'] & M_4
\end{tikzcd}$$

whose image under P is a cartesian square in $\mathbf{A}$, if γ and γ' are cartesian and k_0 is cocartesian, then k_1 is cocartesian.

1.3 Characterization of Descent Data

Assume henceforth that $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ is a Chevalley functor. Let $a : A_1 \to A_0$ be an arrow in $\mathbf{A}$. Let A_2 be the fibred product $A_1 \times_{A_0} A_1$, with canonical projections $a_1, a_2 : A_2 \to A_1$. The property (C) defines, for every object $M_1 \in \mathbf{M}(A_1)$, a canonical bijection, natural in M_1,

$$\mathrm{Hom}_{\mathbf{M}(A_2)}(a_1^*(M_1), a_2^*(M_1)) \to \mathrm{Hom}_{\mathbf{M}(A_1)}(T^a(M_1), M_1),$$

denoted $\varphi \mapsto \mathrm{K}^a(\varphi)$.

Lemma 1. An arrow $\varphi : a_1^*(M_1) \to a_2^*(M_1)$ such that $\mathrm{P}(\varphi) = \mathrm{id}_{A_2}$ is a descent datum if and only if $\mathrm{K}^a(\varphi)$ is an algebra over the monad $\mathbf{T}^a$.

Let $\mathrm{D}(a)$ denote the category of descent data relative to a, and let

$$\Psi^a : \mathbf{M}(A_0) \to \mathrm{D}(a), \quad \mathrm{U}^a : \mathrm{D}(a) \to \mathbf{M}(A_1)$$

be the canonical functors.

Theorem 1. The correspondence $\varphi \mapsto \mathrm{K}^a(\varphi)$ induces an equivalence of categories $\mathrm{K}^a : \mathrm{D}(a) \to \mathbf{M}^a$, making the following diagram commute:

$$\begin{tikzcd}
\mathbf{M}(A_0) \arrow[r, "\Psi^a"] & \mathrm{D}(a) \arrow[r, "\mathrm{K}^a"] \arrow[d, "\mathrm{U}^a"'] & \mathbf{M}^a \arrow[dl, "\mathrm{U}^{\mathbf{T}^a}"] \\
 & \mathbf{M}(A_1) &
\end{tikzcd}$$

Proposition 1. The correspondence $\varphi \mapsto K^{a}(\varphi)$ is universal. Precisely, for an arrow $b_0 : A_0' \to A_0$ in $\boldsymbol{A}$, consider the change-of-base diagram in $\boldsymbol{A}$:

$$
\begin{tikzcd}
A_2' \arrow[r, "a_1'\ \ a_2'"] & A_1' \arrow[r, "b_2\qquad\qquad b_1"] \arrow[dl, "a'"'] & A_2 \arrow[r, "a_1\ \ a_2"] & A_1 \arrow[dl, "a"] \\
A_0' \arrow[rr, "b_0"'] & & A_0 &
\end{tikzcd}
$$

For $M_1 \in \boldsymbol{M}(A_1)$ and $\varphi : a_1^*(M_1) \to a_2^*(M_1)$ in $\boldsymbol{M}(A_2)$,

$$K^{a'}(b_2^*(\varphi)) = b_1^*(K^{a}(\varphi)).$$

In particular, taking $A_0' = A_1$ and $b_0 = a$, if φ is a descent datum, then $b_2^*(\varphi)$ is an effective descent datum. The converse holds, yielding:

Corollary 1. An arrow $\varphi : a_1^*(M_1) \to a_2^*(M_1) \in \boldsymbol{M}(A_2)$ is a descent datum if and only if its inverse image $b_2^*(\varphi)$ under the canonical change of base $b_0 = a : A_0' = A_1 \to A_0$ is an effective descent datum.

This eliminates the need for the "cocycle condition" in subsequent arguments.

2 First Applications

Using Theorem 1, Beck's criterion [2] provides necessary and sufficient conditions for Ψ^{a} to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by a^*.

Proposition 2. If cokernels of pairs of arrows exist in $\boldsymbol{M}(A_0)$, then Ψ^{a} has a left adjoint.

Proposition 3. The functor Ψ^{a} is faithful if and only if a^* is faithful.

Proposition 4. If a^* reflects cokernels, then Ψ^{a} is fully faithful. In particular, if all fibres of $\boldsymbol{M}$ are abelian, then

$$\Psi^{a} \text{ faithful} \iff \Psi^{a} \text{ fully faithful} \iff a^* \text{ faithful}.$$

Definition 2. An arrow $a : A_1 \to A_0$ is *faithfully flat* if a^* commutes with cokernels and reflects isomorphisms.

Proposition 5. If $a : A_1 \to A_0$ is faithfully flat and cokernels exist in $\boldsymbol{M}(A_0)$, then Ψ^{a} is an equivalence of categories.

3 First Examples of Chevalley Functors

1. If A is the dual of the category of commutative rings and M is the dual of the category of modules over varying commutative rings, the obvious functor $\mathrm{P} : \mathsf{M} \to \mathsf{A}$ is Chevalley.

2. If A is a category with fibred products and $\mathsf{M} = \mathsf{Fl}(\mathsf{A})$ is the category of arrows in A, the "target" functor $\mathrm{P} : \mathsf{M} \to \mathsf{A}$ is Chevalley.

3. If $\mathrm{P} : \mathsf{M} \to \mathsf{A}$ and $\mathrm{Q} : \mathsf{N} \to \mathsf{M}$ are Chevalley, their composite $\mathrm{P} \circ \mathrm{Q}$ is Chevalley.

4. If $\mathrm{P} : \mathsf{M} \to \mathsf{A}$ is Chevalley and $\mathbf{I}$ is any category, the functor $\mathrm{P}^{\mathbf{I}} : \mathsf{M}^{\mathbf{I}} \to \mathsf{A}^{\mathbf{I}}$ is Chevalley.

5. In a cartesian diagram of categories

$$\begin{tikzcd} \mathsf{X}' \arrow[r, "\mathsf{f}^*"] \arrow[d, "\mathsf{g}"'] & \mathsf{M} \arrow[d, "\mathrm{P}"] \\ \mathsf{X} \arrow[r, "\mathsf{f}"'] & \mathsf{A} \end{tikzcd}$$

if X has fibred products, f preserves fibred products, and P is Chevalley, then $\mathsf{f}^*(\mathrm{P})$ is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether $\Psi^{\mathfrak{a}}$ is faithful, fully faithful, or an equivalence when the fibres of M are algebraic categories (e.g., categories of modules).

Література

[1] A. Grothendieck, Catégories fibrées et descente, *Séminaire Bourbaki*, 1959.

[2] F. E. J. Linton, Applied functorial semantics II, *Springer Lecture Notes in Mathematics*, No. 80, 1969.

[3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).