Issue XL: Modal Homotopy Type Theory

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Анотація

Formal definition of Cohesive Topos.
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Зміст

1 Modal Homotopy Type Theory

1.1 Preliminaries

A **category** $\mathcal{C}$ consists of:

- A class of **objects**, $\mathrm{Ob}(\mathcal{C})$,
- A class of **morphisms**, $\mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$, for each pair $\mathsf{X}, \mathsf{Y} \in \mathrm{Ob}(\mathcal{C})$,
- Composition maps $\circ : \mathrm{Hom}(\mathsf{Y}, \mathsf{Z}) \times \mathrm{Hom}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Hom}(\mathsf{X}, \mathsf{Z})$,
- Identity morphisms $\mathrm{id}_{\mathsf{X}} \in \mathrm{Hom}(\mathsf{X}, \mathsf{X})$ for each X,

satisfying associativity and identity laws.

A **functor** $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ assigns to each:

- Object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$,

- Morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$,

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A **natural transformation** $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of morphisms $\eta_X : F(X) \to G(X)$ such that for every $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

commutes.

An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of functors

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

and natural transformations (unit η and counit ε)

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F, \quad \varepsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

satisfying the triangle identities.

1.2 Topos

A **topos** $\mathcal{E}$ is a category that:

- Has all finite limits and colimits,

- Is Cartesian closed: has exponential objects $[X, Y]$,

- Has a subobject classifier Ω.

1.3 Geometric Morphism

A **geometric morphism** $f : \mathcal{E} \to \mathcal{F}$ between topoi consists of an adjoint pair

$$f^* : \mathcal{F} \leftrightarrows \mathcal{E} : f_*$$

with $f^* \dashv f_*$, where f^* preserves finite limits (i.e., is left exact).

1.4 Cohesive Topos

A **cohesive topos** is a topos $\mathcal{E}$ equipped with a quadruple of adjoint functors:

$$\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathcal{E} \leftrightarrows \boldsymbol{Set}$$

such that:

- Γ is the global sections functor,
- Δ is the constant sheaf functor,
- ∇ sends a set to a codiscrete object,
- Π is the shape or fundamental groupoid functor,
- Δ and ∇ are fully faithful,
- Δ preserves finite limits,
- Π preserves finite products (in some variants).

1.5 Cohesive Adjunction Diagram and Modalities

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left=3, "\Pi"] \arrow[r, shift left=3, "\Gamma"'] \arrow[r, phantom, "\dashv"] & \boldsymbol{Set} \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, phantom, shift right=3, "\Delta"] \arrow[l, phantom, shift right=3, "\nabla"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "\int"] \arrow[r, bend right, "\sharp"'] \arrow[r, phantom, "\Downarrow \flat"] & \mathcal{E}
\end{tikzcd}$$

1.6 Cohesive Modalities

The above adjoint quadruple canonically induces a triple of endofunctors on $\mathcal{E}$:

$$(\int \dashv \flat \dashv \sharp) : \mathcal{E} \to \mathcal{E}$$

defined as follows:

$$\int \coloneqq \Delta \circ \Pi$$
$$\flat \coloneqq \Delta \circ \Gamma$$
$$\sharp \coloneqq \nabla \circ \Gamma$$

This yields an **adjoint triple** of endofunctors on $\mathcal{E}$:

$$\int \dashv \flat \dashv \sharp$$

These are:

- $\int$ — the **shape modality**: captures the fundamental shape or homotopy type,
- $\flat$ — the **flat modality**: forgets cohesive structure while remembering discrete shape,
- $\sharp$ — the **sharp modality**: codiscretizes the structure, reflecting the full cohesion.

Each of these is an **idempotent** (co)monad, hence a *modality* in the internal language (type theory) of $\mathcal{E}$.

1.7 Differential Cohesion

A **differential cohesive topos** is a cohesive topos $\mathcal{E}$ equipped with an additional adjoint triple of endofunctors:

$$(\mathfrak{R} \dashv \mathfrak{I} \dashv \&) : \mathcal{E} \to \mathcal{E}$$

These are:

- $\mathfrak{R}$: the **reduction modality** — forgets nilpotents,
- $\mathfrak{I}$: the **infinitesimal shape modality** — retains infinitesimal data,
- $\&$: the **infinitesimal flat modality** — reflects formally smooth structure.

Important object classes:

- An object X is **reduced** if $\mathfrak{R}(\mathsf{X}) \cong \mathsf{X}$.
- It is **coreduced** if $\&(\mathsf{X}) \cong \mathsf{X}$.
- It is **formally smooth** if the unit map $\mathsf{X} \to \&\mathsf{X}$ is an effective epimorphism.

Formally étale maps are those morphisms $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ such that the square

$$\begin{tikzcd} \mathsf{X} \arrow[r] \arrow[d, "\mathsf{f}"'] & \mathfrak{I}\mathsf{X} \arrow[d, "\mathfrak{I}(\mathsf{f})"] \\ \mathsf{Y} \arrow[r] & \mathfrak{I}\mathsf{Y} \end{tikzcd}$$

is a pullback.

1.8 Graded Differential Cohesion

In **graded differential cohesion**, such as used in synthetic supergeometry, one introduces an adjoint triple:

10) $\rightrightarrows \dashv \rightsquigarrow \dashv \mathrm{Rh}$

$$(\rightrightarrows \dashv \rightsquigarrow \dashv \mathrm{Rh}) : \mathcal{E} \to \mathcal{E}$$

These are:

- $\rightrightarrows$: the **fermionic modality** — captures anti-commuting directions,
- $\rightsquigarrow$: the **bosonic modality** — filters out fermionic directions,
- Rh: the **rheonomic modality** — encodes constraint structures.

These modal operators form part of the internal logic of supergeometric or supersymmetric type theories.

1.9 Adjoint String of Identity Modalities

In Homotopy Type Theory (HoTT), identity systems (Contractible, Strict Id, Quotient, Isomorphism, Path = Equivalence) are modeled as modalities in the ∞-topos $\mathcal{E} = \infty$Grp. We construct an adjoint quadruple extending the Jacobs-Lawvere triple $\mathsf{C} \dashv \mathrm{Id}_A \dashv \mathsf{Q}(-/\sim)$, incorporating Isomorphism and Path = Equivalence. The modalities are ordered by adjointness: Contractible $\leq$ Strict Id $\leq$ Quotient $\leq$ Isomorphism $\leq$ Path = Equivalence, reflecting their structure in HoTT, where Strict Id, Quotient, and Path = Equivalence are mere propositions for h-sets, while Isomorphism is not.

Homotopy Type Theory (HoTT) provides a framework for reasoning about equality via the identity type $\mathrm{Id}_A(x, y)$. In the ∞-topos $\mathcal{E} = \infty$Grp, identity systems are modalities (monads), ordered by adjointness. The classical Jacobs-Lawvere adjunction triple $\mathsf{C} \dashv \mathrm{Id}_A \dashv \mathsf{Q}(-/\sim)$ captures **Contractible**, **Strict**, and **Quotient**. We extend this to a quadruple, including **Isomorphism** and **Path = Equiv**, respecting the HoTT equivalence of Path and Equivalence and the propositional nature of Strict Id, Quotient, and Path = Equivalence for h-sets.

Definition 1. In HoTT, the identity systems are:

- **Contractible**: (-1)-truncated types, mere propositions.
- **Strict**: $\mathrm{Id}_A(x, y)$ for h-sets (0-truncated), a mere proposition.
- **Quotient**: Set-quotients $A/\sim$, 0-truncated, equivalent to Strict Id.
- **Isomorphism**: $\mathrm{Iso}_A(x, y)$, a triple (f, g, p), not a mere proposition.
- **Path = Equiv**: $\mathrm{Path}_A(x, y) \simeq (x \simeq y)$, equivalent in HoTT.

In $\mathcal{E} = \infty$Grp, we define categories:

- $\mathcal{E}_{\text{contr}} = \mathcal{E}_{\leq -1}$: Mere propositions.
- $\mathcal{E}_{\text{strict}} = \mathcal{E}_{\leq 0} \cong$ Set: h-sets (Strict Id).
- $\mathcal{E}_{\text{quot}} = \mathcal{E}_{\leq 0} \cong$ Set: h-sets (Quotient).
- $\mathcal{E}_{\text{iso}} \cong \mathcal{E}$: ∞-groupoids with isomorphisms.
- $\mathcal{E}_{\text{path/equiv}} \cong \mathcal{E}$: ∞-groupoids with paths/equivalences.

The Jacobs-Lawvere triple $C \dashv \mathrm{Id}_A \dashv Q(-/\sim)$ is extended to an adjoint quadruple:

$$\mathcal{E}_{\text{contr}} \xrightarrow{F_4} \mathcal{E}_{\text{strict}} \xrightarrow{F_3} \mathcal{E}_{\text{quot}} \xrightarrow{F_2} \mathcal{E}_{\text{iso}} \xrightarrow{F_1} \mathcal{E}_{\text{path/equiv}}$$

Theorem 1. The functors form an adjoint quadruple with adjunctions:

$$F_4 \dashv U_4, \quad F_3 \dashv U_3, \quad F_2 \dashv U_2, \quad F_1 \dashv U_1$$

- $F_4 : \mathcal{E}_{\textbf{contr}} \to \mathcal{E}_{\textbf{strict}}$: Inclusion of (-1)-truncated objects into 0-truncated objects. Right adjoint U_4: (-1)-truncation, $U_4(X) = \|X\|_{-1}$.
- $F_3 : \mathcal{E}_{\textbf{strict}} \to \mathcal{E}_{\textbf{quot}}$: Canonical map to quotient structure, viewing h-sets as quotiented by trivial relations. Right adjoint U_3: Inverse map preserving h-set structure.
- $F_2 : \mathcal{E}_{\textbf{quot}} \to \mathcal{E}_{\textbf{iso}}$: Inclusion of h-sets into $\mathcal{E}$, $\mathrm{core}(X) \cong X$. Right adjoint U_2: 0-truncation, $U_2(X) = \|X\|_0$.
- $F_1 : \mathcal{E}_{\textbf{iso}} \to \mathcal{E}_{\textbf{path/equiv}}$: Canonical inclusion of ∞-groupoids with isomorphisms into full ∞-groupoids with paths/equivalences. Right adjoint U_1: Core map, preserving isomorphism structure.

The adjunctions induce the ordering:

$$\text{Contractible} \leq \text{Strict Id} \leq \text{Quotient} \leq \text{Isomorphism} \leq \text{Path} = \text{Equivalence}$$

- **Contractible**: Coarsest, mere propositions ((-1)-truncated).
- **Strict**: h-sets, $\mathrm{Id}_A(x, y)$ is a mere proposition.
- **Quotient**: Equivalent to Strict Id, 0-truncated set-quotients.
- **Isomorphism**: $\mathrm{Iso}_A(x, y)$ is not a mere proposition for general types.
- **Path = Equivalence**: Finest, full ∞-groupoid structure, equivalent via univalence.

The adjoint quadruple extends the Jacobs-Lawvere triple, capturing the structure of identity systems in HoTT. The ordering reflects their increasing complexity, with Strict Id, Quotient, and Path = Equivalence collapsing to mere propositions for h-sets, while Isomorphism retains higher structure. Future work could explore these adjunctions in other ∞-topoi or specific CTT models.