Issue XLVII: Transpension and Tiny Amazing Right Adjoints

Namdak Tonpa

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Анотація

We present a detailed type-theoretic and categorical exposition of modal type theory, transpension types, and tiny objects. In category theory, a tiny object $\mathbb{U}$ has the property that the exponential functor $(-)^{\mathbb{U}}$ has a right adjoint, known as the "amazing right adjoint."In modal dependent type theory, this adjoint structure is internalized via the transpension type former, which acts as the right adjoint to universal quantification over the shape $\mathbb{U}$. We detail the syntactic rules, the transposition bijection, and show how the transpension type former unifies various presheaf internalization operators such as Glue, Weld, and Gel.

Keywords: Category Theory, Topos Theory, Modal HoTT, Transpension Types

Зміст

1 Introduction

Modern type theories, such as Homotopy Type Theory (HoTT) and Simplicial Homotopy Type Theory (STT), are designed to reason about mathematical objects synthetically. However, standard type systems are limited to constructions that are equivariant with respect to variables in the context. In presheaf models, this manifests as the requirement that all operations must be compatible with the base category's coordinate changes.

To break this limitation internally, we require modal operations. Modal dependent type theory (MTT) extends the syntax to reason about sub-structural transformations (such as taking the core groupoid of an ∞-category or restricting to constant/discrete shapes). The ultimate realization of this framework is the *transpension type*, which internalizes the category-theoretic notion of tiny objects and Lawvere's amazing right adjoint.

2 Tiny Objects and the Amazing Right Adjoint

In a Cartesian closed category (or topos) $\mathcal{E}$, the product functor $- \times \mathsf{U}$ has a right adjoint, namely the exponential functor $(-)^{\mathsf{U}}$:

$$- \times \mathsf{U} \dashv (-)^{\mathsf{U}}$$

This adjunction corresponds syntactically to the bijection between functions of two variables and curried functions.

Definition 1 (Tiny Object)**.** An object $\mathsf{U} \in \mathcal{E}$ is called **tiny** (or **atomic**) if the exponentiation functor $(-)^{\mathsf{U}} : \mathcal{E} \to \mathcal{E}$ itself possesses a right adjoint, denoted as $\sqrt[\mathsf{U}]{-}$ or $(-)^{1/\mathsf{U}}$:

$$(-)^{\mathsf{U}} \dashv \sqrt[\mathsf{U}]{-}$$

This results in an adjoint string:

$$(-) \times \mathsf{U} \quad \dashv \quad (-)^{\mathsf{U}} \quad \dashv \quad \sqrt[\mathsf{U}]{-}$$

The functor $\sqrt[\mathsf{U}]{-}$ is called the **amazing right adjoint** to the exponential functor.

Example 1 (Infinitesimals in SDG)**.** In Lawvere's Synthetic Differential Geometry (SDG), the object of infinitesimals $\mathsf{D} = \{x \in \mathsf{R} \mid x^2 = 0\}$ is tiny. The existence of the amazing right adjoint to $(-)^{\mathsf{D}}$ (the tangent bundle functor) allows for the internalization of differential forms and smooth flows.

Example 2 (Intervals in Cubical and Simplicial HoTT)**.** In cubical type theory (such as models based on Dedekind cubes), the interval object I is tiny, yielding the amazing right adjoint $\sqrt[\mathrm{I}]{-}$. This category-theoretic property provides a clean syntactic solution to the definition of transport (`transp`) and composition (`hcomp`) operations. In standard cubical type theory (e.g., the CCHM model), these operators must be introduced as primitive constants for each type former (functions, pairs, universes, etc.), leading to a highly complex set of syntax rules and reduction relations.

However, if I is tiny, the dependent amazing right adjoint (the transpension type $\mathrm{Transpension}_{\mathrm{I}}$) allows one to define fibrancy (the Kan composition and transport structures) internally. Specifically, the condition that a type family A over I has transport corresponds to the existence of a diagonal filler in the lifting square:

$$\begin{array}{ccc} \partial \mathrm{I} & \longrightarrow & \Sigma(\mathrm{I}, A) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{I} & \xrightarrow[\mathrm{id}_{\mathrm{I}}]{} & \mathrm{I} \end{array}$$

which can be represented internally using the transpension type over the boundary $\partial \mathrm{I} \subset \mathrm{I}$. Furthermore, the cubical `Glue` type former—which constructs paths from equivalences and is necessary to prove the univalence axiom—can be constructed internally. Given an equivalence $e : B \simeq A$, the `Glue` type is defined by transposing e along the inclusion of the boundary $\partial \mathrm{I} \to \mathrm{I}$ using the amazing right adjoint.

In simplicial type theory (SHTT), the simplicial interval $\mathbb{I}^{\rightarrow}$ is not tiny in the standard model of simplicial sets, which prevents the internal definition of the universe of $(\infty, 1)$-categories. By moving to bisimplicial sets or cubical spaces where $\mathbb{I}^{\rightarrow}$ is tiny, or by introducing modal extensions as in the works of Gratzer, Weinberger, and Buchholtz, the amazing right adjoint is restored, enabling the internalization of the Yoneda lemma and directed univalence.

3 Modal Dependent Type Theory

Modal dependent type theory (MTT) formalizes these adjoint structures syntactically by modifying both contexts and types.

3.1 Context Restrictions and Transposition

Instead of only modifying types, MTT introduces **context restrictions**. For a modality μ corresponding to an adjoint pair $L \dashv R$, we define a context restriction operator $\Gamma.\{\mu\}$ representing the application of the left adjoint L to the context.

Correspondingly, we define the **modal type** $\langle \mu \mid A \rangle$ representing the right adjoint R. The fundamental syntax rule of MTT is the **transposition bijection** on term sets:

$$\mathrm{Tm}(\Gamma.\{\mu\}, A) \cong \mathrm{Tm}(\Gamma, \langle \mu \mid A \rangle)$$

Under this bijection, a term $t : \langle \mu \mid A \rangle$ in the normal context Γ is transposed to a term $\widetilde{t} : A$ in the restricted context $\Gamma.\{\mu\}$.

3.2 Elimination and Modal Dependent Products

To eliminate a modal type $\langle \mu \mid A \rangle$, we use the let-binding construct:

$$\text{let } \mathrm{mod}_{\mu}(x) \leftarrow t \text{ in } u$$

where $t : \langle \mu \mid A \rangle$ and $x : A$ is bound in u. The variable x is only accessible in modal-restricted parts of u.

We can also form the **modal dependent product** (or modal Π-type):

$$(x :_\mu A) \to B(x)$$

which classifies functions whose domain is restricted by the modality μ. Semantically, this corresponds to the dependent right adjoint to the modal context extension.

4 The Transpension Type

Introduced by Andreas Nuyts and Dominique Devriese, the **transpension type** former internalizes the amazing right adjoint within dependent type theory.

Let $\mathbb{U}$ be a shape (a tiny object). Exponentiation by $\mathbb{U}$ is syntactically represented by the universal quantifier or dependent function type $\forall(u : \mathbb{U}), -$. The transpension type is defined as the dependent right adjoint:

$$\forall(u : \mathbb{U}) \quad \dashv \quad \text{transpension}_{\mathbb{U}}$$

4.1 Rules for Transpension

The transpension type former associates to each type family A over $\mathbb{U}$ a type $\text{Transpension}_{\mathbb{U}}(A)$ in the base context:

- **Formation**:
$$\frac{\Gamma, u : \mathbb{U} \vdash A(u) : \text{Type}}{\Gamma \vdash \text{Transpension}_{\mathbb{U}}(A) : \text{Type}}$$
- **Introduction**:
$$\frac{\Gamma, u : \mathbb{U} \vdash t(u) : A(u)}{\Gamma \vdash \text{transp}(t) : \text{Transpension}_{\mathbb{U}}(A)}$$
- **Elimination**: The elimination rule allows us to project out the dependent family:
$$\frac{\Gamma \vdash p : \text{Transpension}_{\mathbb{U}}(A) \quad \Gamma, u : \mathbb{U} \vdash B(u) : \text{Type}}{\Gamma, u : \mathbb{U} \vdash \text{proj}(p, u) : A(u)}$$
- **Computation**:
$$\text{proj}(\text{transp}(t), u) \equiv t(u)$$
- **Uniqueness**:
$$\text{transp}(\lambda u.\text{proj}(p, u)) \equiv p$$

Semantically, this dependent adjunction mirrors the category-theoretic transposition $\text{Hom}(X, Y^{\mathbb{U}}) \cong \text{Hom}(X \times \mathbb{U}, Y)$ in the slice category over the base, extending Lawvere's amazing right adjoint to dependent type families.

5 Unification of Presheaf Internalizations

The primary significance of the transpension type is its expressive power: it serves as a unifying primitive that recovers several other ad-hoc operators used to internalize presheaf semantics:

1. **Glue Types**: In Cubical HoTT (such as the CCHM model), for a cofibration φ and an equivalence $e : B \simeq A$ defined under φ, one forms the type:

   $$\text{Glue}[\varphi \mapsto (B, e)]A$$

   When we set the shape to be the cubical interval I, with $V := \partial I \subset I$ as the boundary, the equivalence e can be seen as a dependent family on V. Using the transposition bijection of the transpension type, the equivalence e is transposed over the boundary map $\partial I \hookrightarrow I$ using the amazing right adjoint. The pullback of this transposed equivalence yields a type family over I whose fibers are B over ∂I and A elsewhere. The transpension type $\text{Transpension}_I(e)$ thus directly acts as the "glued"type representing the path of types, generalizing and recovering the primitive Glue type constructor without requiring it as an ad-hoc syntactic rule.

2. **Gel Types (Ψ)**: In relationally parametric type theories (e.g., Cavallo and Harper's model), given two types A, B and a relation $R : A \to B \to \text{Type}$, the Gel type:

   $$\text{Gel}_R(A, B)$$

   constructs a "bridge"type. If we define the shape U to be the "bridge interval"I_{bridge} with boundary $\partial I_{\text{bridge}} := \{0, 1\}$, a type family $C(u)$ over I_{bridge} corresponds to the pair of types $C(0) = A$ and $C(1) = B$, and the relation R. The transpension type $\text{Transpension}_{I_{\text{bridge}}}(C)$ internalizes the relation R by mapping it to a type whose terms are pairs of elements $(a : A, b : B)$ satisfying the relation R. This is exactly the relational Gel type, showing that transpension over the bridge shape recovers relationally parametric models.

3. **Weld Types**: In nominal and guarded type theories, the Weld type constructor:

   $$\text{Weld } n\ A$$

   is used to bind a fresh name n and identify terms at specific boundaries. By letting the shape U be the set of nominal names (or a name space), the transpension type former $\text{Transpension}_{\text{Name}}(A)$ acts as the right adjoint to quantification over names, which represents nominal binding. By transposing terms across the name-space shape, transpension recovers the nominal Weld operation, allowing one to identify terms at name boundaries and handle fresh name generation internally.

This unification establishes a precise dictionary between geometric modalities and internal presheaf operators:

- **Synthetic Topology** $\longleftrightarrow$ **Glue Types**: Glues homotopical spaces along cubical boundaries using the interval I, lifting equivalences to paths to construct the univalent universe of spaces.
- **Directed Category Theory** $\longleftrightarrow$ **Gel Types**: Formulates directed hom-types ($x \to y$) and relational parametricity over directed or bridge intervals (I_{bridge}), internalizing relations as directed path shapes.
- **Differential Cohesion** $\longleftrightarrow$ **Weld Types**: Resolves formal neighborhoods and infinitesimal displacements. Infinitesimals function as bound nominal names or variables; the Weld operator identifies points that are formally close (infinitesimally close) to one another.

By integrating transpension with strictness axioms and Multimodal Type Theory, we obtain a consistent syntactic framework for synthetic topology, differential cohesion, and directed category theory.

Література

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